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SOLVABILITY OF DUFFING EQUATIONS AT RESONANCE

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Abstract

We give new existence conditions of solutions for Duffing equations at resonance using some results in [2].

1. Introduction and Main Results

Consider the BVP at resonance

$$x'' + k^2\pi^2 x + g(t, x) + h(t, x) = 0, \tag{1.1}$$

$$x(0) = 0 = x(1), \tag{1.2}$$

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where $g, h : [0, 1] \times \mathbf{R} \to \mathbf{R}$ are Caratheodory functions, k is a fixed natural number. There exists $\overline{h} \in L^{\infty}[0, 1]$ such that

$$|h(t, x)| \leq \overline{h}(t), \text{ for a.e. } t \in [0, 1], \ x \in \mathbf{R}. \tag{1.3}$$

And we will use the following conditions:

(L_1) There exists a constant $r_0 > 0$ and $q \in L^{\infty}[0, 1]$ such that

$$\limsup_{x \to \infty} g(t, x) / x \leq q(t), \tag{1.4}$$

uniformly for a.e. $t \in [0, 1]$, and

$$g(t, x)x \geq 0 \text{ for } |x| \geq r_0, \text{ a.e. } t \in [0, 1]. \tag{1.5}$$

(L_2)

$$\int_{v>0} f_+(t)v(t)dt + \int_{v<0} f_-(t)v(t)dt > 0, \tag{1.6}$$

where $v = \pm \sin k\pi t$, $t \in [0, 1]$, $f = g + h$, $f_+(t) = \liminf_{x \to +\infty} f(t, x)$, $f_-(t) = \limsup_{x \to -\infty} f(t, x)$.

As in [2], we write $b \in H_n(1, (0, 1), 0, \pi)$, if and only if the problem

$$x'' + b(t)x = 0, \tag{1.7}$$

$$x(0) = 0 = x(1)$$

has a nontrivial solution with exactly n zeros on $(0, 1)$. The main result of this paper is the following:

Theorem 1. *Assume that* $g, h : [0, 1] \times \mathbf{R} \to \mathbf{R}$ *are Caratheodory functions such that conditions* (1.3), (L_1), *and* (L_2) *are valid. Then* (1.1-1.2) *has at least one solution provided that*

$$k^2\pi^2 + q < q_k \tag{1.8}$$

for some $q_k \in H_k(1, (0, 1), 0, \pi)$.

This result is inspired and improves by Kuo [5], who showed by making use of a Lyapunov type inequality and the well-known Leray-Schauder continuation method, that (1.1-1.2) has at least one solution if (1.3), (L_1), and (L_2) are satisfied and

$$\int_0^1 q(t)dt < 2k(k+1)\pi \tan \frac{\pi}{2(k+1)}. \tag{1.9}$$

We will show that (1.9) is a special case of (1.8). In fact, we prove that if

$$q \geq A - k^2\pi^2, \int_0^1 q(t)dt < A - k^2\pi^2 + 2\sqrt{A}(k+1)\tan \frac{(k+1)\pi - \sqrt{A}}{2(k+1)}, \tag{1.10}$$

where $A \in (k^2\pi^2, (k+1)^2\pi^2)$, then (1.8) is satisfied. And (1.9) is a special case of (1.10) as $A \to k^2\pi^2 + 0$. Moreover, some sufficient conditions of Theorem 1 will be given such that $\int_0^1 q(t)dt$ could be large enough with k fixed, and hence they could be applied to some new cases.

We finally emphasize that condition (1.8) is not only a sufficient one but also a necessary one, that is, for g, h satisfying (1.3), (L_1), and (L_2) (1.1-1.2) has at least one solution, if and only if (1.8) is satisfied. In addition, condition (1.8) is not more difficult to verify than (1.9). In fact, we only need to estimate the value at $t = 1$ of the unique solution of the problem (2.1-2.2) in the next section.

2. Proof of Theorem 1

In this section, we will finish the proof of Theorem 1. To this end for $b \in L^\infty[0, 1]$ we consider the initial value problem

$$\phi' = \cos^2 \phi + b(t)\sin^2 \phi, \tag{2.1}$$

$$\phi(0) = 0. \tag{2.2}$$

Denote the unique solution of (2.1-2.2) by $\phi = \phi(t, b)$.

Lemma 1. $b \in H_n(1, (0, 1), 0, \pi)$, *if and only if* $\phi(1, b) = (n+1)\pi$.

Proof. Refer to [4, Chapter 11, Lemma 3.1].

Lemma 2. *Assume that* $\{q_n\} \subset L^{\infty}[0, 1]$, $Q \in L^{\infty}[0, 1]$ *are such that* $|q_n(t)| \leq Q(t)$ *for a.e.* $t \in [0, 1]$, $n = 1, 2, 3, \cdots$, *and* $q_n(t) \to q(t)$ *for a.e.* $t \in [0, 1]$. *Then* $\phi(t, q_n) \to \phi(t, q)$ *uniformly for* $t \in [0, 1]$.

Proof. Refer to [4, p.4, Theorem 2.4].

Lemma 3. *Assume that* $q_1, q_2 \in L^{\infty}[0, 1]$ *are such that* $q_1 \leq q_2$. *Then* $\phi(t, q_1) \leq \phi(t, q_2)$ *for* $t \in [0, 1]$. *In addition, assume that* $q_1 < q_2$. *Then we have* $\phi(1, q_1) < \phi(1, q_2)$.

Proof. Refer to [4, Corollary 4.2] or [2, Lemma 5].

Remark 1. From Lemmas 1, 3, and [4, Chapter 8, Theorem 2.1] it follows that (1.8) is equivalent to

$$\phi(1, q + k^2\pi^2) < (k+1)\pi.$$

Lemma 4. *Assume that* (1.8) *is satisfied with* $q \geq 0$. *Then there exists* $\epsilon > 0$ *such that for every* $b \in L^{\infty}[0, 1]$ *with* $k^2\pi^2 < b < q + k^2\pi^2 + 2\epsilon$, (1.7-1.2) *has only the trivial solution.*

Proof. By Lemmas 1, 3, and condition (1.8), we have

$$\phi(1, k^2\pi^2 + q) < \phi(1, q_k) = (k+1)\pi.$$

From Lemma 2 there exists $\epsilon > 0$ such that

$$\phi(1, k^2\pi^2 + q + 2\epsilon) < (k+1)\pi.$$

Therefore, for every $b \in L^{\infty}[0, 1]$ with $k^2\pi^2 < b < k^2\pi^2 + q + 2\epsilon$, we have from Lemmas 1, 3 that

$$k\pi = \phi(1, k^2\pi^2) < \phi(1, b) < \phi(1, k^2\pi^2 + q + 2\epsilon) < (k+1)\pi.$$

And hence, the conclusion follows from Lemma 1. ∎

Lemma 5. *Assume that* $\{y_n\} \subset W_0^{2,1}[0, 1]$, $y_n(t) \to \sin k\pi t (in C^1 [0, 1])$. *Then there exists* $\{u_n\} \subset W_0^{2,1}[0, 1]$, *such that* $u_n(t) \to \sin k\pi t$ *in* $C^1[0, 1]$,

$$u_n(t) y_n(t) \geq 0 \text{ for } t \in [0, 1], \ n \text{ large enough},$$

and

$$\int_0^1 u_n(t)[y_n''(t) + k^2\pi^2 y_n(t)]dt = 0, \ n \text{ large enough}.$$

Proof. Directly follows from [1, pages 412-413].

Proof of Theorem 1. In view of the Leray-Schauder principle, we only need to show that the possible solutions of the following auxiliary problem are a priori bounded:

$$x'' + k^2\pi^2 x + \lambda \epsilon x + (1 - \lambda)[g(t, x) + h(t, x)] = 0, \ \lambda \in (0, 1),$$

$$x(0) = 0 = x(1).$$

By contradiction, suppose that $\{x_n\} \subset W_0^{2,1}[0, 1]$ with $|x_n|_1 \to +\infty$, $\{\lambda_n\} \subset (0, 1)$ are such that

$$x_n'' + k^2\pi^2 x_n + \lambda \epsilon x_n + (1 - \lambda)[g(t, x_n) + h(t, x_n)] = 0, \tag{2.3}$$

$$x_n(0) = 0 = x_n(1). \tag{2.4}$$

From (L_1) there exists $r_1 \geq r_0$ such that

$$g(t, x)/x \leq q(t) + \epsilon \tag{2.5}$$

for a.e. $t \in (0, 1)$, $x \in \mathbf{R}$ with $|x| \geq r_1$.

Set

$$y_n(t) = x_n(t)/|x_n|_1, \tag{2.6}$$

$$\mu_n(t) = g(t, x_n(t))/x_n(t) \text{ as } |x_n(t)| \geq r_1 \tag{2.7}$$

$$= 0 \text{ as } |x_n(t)| < r_1, \tag{2.8}$$

and

$$h_n(t) = g(t, x_n(t)) + h(t, x_n(t)) - \mu_n(t).$$

From (1.3) and (2.5) we have

$$0 \le \mu_n \le q + \epsilon, \tag{2.9}$$

$$|h_n(t)| \le \overline{h}(t) \text{ for a.e. } t \in (0, 1),\ n = 1, 2, 3, \cdots, \tag{2.10}$$

where $\overline{h} \in L^1[0, 1]$, and (2.3-2.4) is equivalent to

$$y_n'' + k^2\pi^2 y_n + \lambda_n \epsilon y_n + (1 - \lambda_n)\mu_n(t)y_n + (1 - \lambda_n)|x_n|_1^{-1} h_n(t) = 0, \tag{2.11}$$

$$y_n(0) = 0 = y_n(1). \tag{2.12}$$

By (2.6-2.7), [3, Theorem 8.8] and Ascoli-Arzela's theorem, we can assume by going to subsequence if necessary that $\mu_n \rightharpoonup \mu_0$ in $L^\infty[0, 1]$ with $0 \le \mu_0 \le q + \epsilon$, $y_n \to y_0$ in $C^1[0, 1]$ and $\lambda_n \to \lambda_0 \in [0, 1]$. Integrating (2.8) over $[0, t]$ for every $t \in (0, 1]$ and taking the limit as $n \to \infty$, we have

$$y_0'' + [k^2\pi^2 + \lambda_0 \epsilon + (1 - \lambda_0)\mu_0(t)]y_0 = 0,$$

$$y_0(0) = 0 = y_0(1).$$

From Lemma 3 we have that $\lambda_0 = 0$, $\mu_0 = 0$, and

$$y_0'' + k^2\pi^2 y_0 = 0,$$

$$y_0(0) = 0 = y_0(1).$$

Since $y_n \to y_0$ by Lemma 5 there exists u_n such that $u_n \to y_0$ in $C^1[0, 1]$, $u_n(t)y_n(t) \ge 0$ for $t \in [0, 1]$, n large enough and

$$\int_0^1 u_n(t)[y_n''(t) + k^2\pi^2 y_n(t)]dt = 0, \ n \text{ large enough.}$$

Noticing that $f = g + h$, from (2.3) and (L_2) we have

$$\int_0^1 f(t, x_n(t))u_n(t)dt = \int_0^1 [g(t, x_n(t)) + h(t, x_n(t))]u_n(t)dt \le 0, \ n$$

large enough.

In view of (1.3-1.4) there exists $F \in L^1[0, 1]$ such that

$$f(t, x_n(t)) \ge F(t), \text{ for a.e. } t \in [0, 1], \ n \text{ large enough.}$$

Using Fatou's lemma, we have

$$\int_{y_0 > 0} f_+(t)y_0(t)dt + \int_{y_0 < 0} f_-(t)y_0(t)dt \le \liminf_{n\to\infty} \int_0^1 f(t, x_n(t))u_n(t)dt \le 0,$$

a contradiction to (1.6). The proof is complete. ■

3. Examples

In this section, we will give some applications of Theorem 1.

Proposition 1. *Assume that* $\{t_i\}_{i=0}^{2n} \subset [0, 1]$ *with* $0 = t_0 < t_1 < \cdots < t_{2n} = 1$. *Set*

$$\overline{q}(t) = \frac{\pi^2}{4(t_i - t_{i-1})^2} \text{ as } t \in (t_{i-1}, t_i), \ i = 1, 2, \cdots, 2n.$$

Then $\overline{q} \in H_{n-1}(1, (0, 1), \pi)$.

Proof. When $n = 1$ let

$$\overline{u}(t) = \sin \frac{\pi(t - t_0)}{2(t_1 - t_0)} \text{ for } t \in [t_0, t_1] \tag{3.1}$$

$$= \cos \frac{\pi(t - t_1)}{2(t_2 - t_1)} \text{ for } t \in [t_1, t_2], \tag{3.2}$$

and when $n \ge 2$, let

$$\overline{u}(t) = \sin \frac{\pi(t - t_0)}{2(t_1 - t_0)} \text{ for } t \in [t_0, t_1]$$

$$= \cos\frac{\pi(t - t_1)}{2(t_2 - t_1)} \text{ for } t \in [t_1, t_2]$$

$$= \left(\prod_{i=1}^{j}\frac{t_{2i+1} - t_{2i}}{t_{2i} - t_{2i-1}}\right)\sin\frac{\pi}{2}\left(2j + \frac{t - t_{2j}}{t_{2j+1} - t_{2j}}\right) \text{ for } t \in [t_{2j}, t_{2j+1}],$$

$$= \left(\prod_{i=1}^{j}\frac{t_{2i+1} - t_{2i}}{t_{2i} - t_{2i-1}}\right)\cos\frac{\pi}{2}\left(2j + \frac{t - t_{2j+1}}{t_{2j+2} - t_{2j+1}}\right) \text{ for } t \in [t_{2j+1}, t_{2j+2}],$$

$$j = 1, 2, \cdots, n-1.$$

It is easy to verify that $\overline{u}$ has n–1 zeros on $(0, 1)$ and

$$\overline{u}''(t) + \overline{q}(t)\overline{u}(t) = 0 \text{ for } t \in (t_i, t_{i+1}),\ i = 1, 2, \cdots, 2n - 1.$$

$$\overline{u}(0) = \overline{u}(1) = 0.$$

Therefore, $\overline{q} \in H_{n-1}(1, (0, 1), \pi)$.

Remark 2. It is easily seen that

$$\int_0^1 \overline{q}(t)dt = \left(\sum_{i=1}^{2n}\frac{1}{t_i - t_{i-1}}\right)\frac{\pi^2}{4} > \frac{1}{t_1} \to +\infty \text{ as } t_1 \to 0^+.$$

Therefore, we could give some new existence results for (1.1-1.2) compared with (1.9).

Proposition 2. *Assume that* $q \in L^\infty[0, 1]$ *satisfies* (1.10) *for some constant* $A \in (k^2\pi^2, (k+1)^2\pi^2)$. *Then* q *satisfies* (1.8) *for some* $q_k \in H_k(1, (0, 1), 0, \pi)$.

Proof. From [6, Theorem 1.3], if q satisfies (1.10), then (1.7-1.2) has only the trivial solution for $b \in L^\infty[0, 1]$ with $A \le b \le k^2\pi^2 + q$. Since $A \in (k^2\pi^2, (k+1)^2\pi^2)$, we have $\phi(1, A) \in (k\pi, (k+1)\pi)$. By Lemmas 1-3, we have that $\phi(1, k^2\pi^2 + q) < (k+1)\pi$. The proof is complete. ■

Remark 3. Let $\psi(A) = A - k^2\pi^2 + 2\sqrt{A}(k+1)\tan\dfrac{(k+1)\pi - \sqrt{A}}{2(k+1)}$.

We have for $A > k^2\pi^2$ that

$$\psi(A) > \psi(k^2\pi^2) = 2k(k+1)\pi\tan\frac{\pi}{2(k+1)}.$$

Therefore, if (1.9) is valid, then there exists $\epsilon > 0$ such that

$$\int_0^1 [q(t) + \epsilon]dt < \psi(k^2\pi^2) < \psi(\epsilon + k^2\pi^2).$$

That is, (1.10) is satisfied. And hence (1.9) is a special case of (1.8).

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